

Classical and Quantum Quintessence Cosmology

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This paper implements the idea of considering the instantonic creation of brane worlds whose five-dimensional bulk contains a negative cosmological constant and a scalar quintessence field with time-dependent equation of state, restricting to the case that the quintessence field couples minimally to Hilbert-Einstein gravity. We construct an Euclidean formalism, both for the four- and five-dimensional cases, singling out a Hamiltonian constraint that depends on the parameter defining the quintessence state equation. Specializing at several particular values of that parameter, we obtain solutions to the constraint equation and analyse them both classically and quantum mechanically. It is found that these solutions can represent either asymptotically anti-de Sitter wormholes or pure anti-de Sitter spaces whose quantum states are obtained by means of the Wheeler de Witt equation. Starting with the different five-dimensional solutions, an instantonic procedure is applied to describe the creation of geometrically equivalent inflating de Sitter branes whose quantum states are also evaluated in some cases. We thus consider the quantum state of the universe to be contributed by all the instantonic paths that correspond to these particular brane worlds.

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I. INTRODUCTION

In the opinion of many cosmologists, recent observations have actually promoted cosmology up to the status of a precision science. In particular, the measurements made on distant supernovas Ia [1] and posterior observations and data refinements [2] have led to the increasingly firmer conclusion that our universe is now accelerating. On the other hand, the even more recent balloon-type experiments [3,4] have quite nicely fixed the spectrum of CMB anisotropies, at least for the initial region of small angles. Although these observations should be extended to even larger redshifts and smaller angles, such as several missions, currently under preparation, will do in the near future [5,6], it already appears well-proved that around two-third of the energy in the universe should be hidden as a dark energy [7]. The most successful and general way to account for this dark energy has been suggested to be the incorporation of a cosmological, so-called quintessence field [8] which advantageously replaces the cosmological constant and can be nicely fitted to predict both cosmic acceleration and a suitable spectrum of CMB anisotropies [9].

Quintessence was first conceived [8] as a slowly-varying, scalar field with constant equation of state, $\omega = \text{Const.} < 0$ (in $p = \omega\rho$), but it soon appeared that in order for this field to allow cosmological scenarios more adaptable to the foreseeable demands of future observations and more compatible with possibly related particle physics theories, one had to endow the quintessence field with the additional degrees of freedom resulting from a time-variable equation of state and a larger

interval for ω (i.e. the quintessence field is no longer necessarily slowly-varying or confined to satisfy a constant value for ω). Tracking models of quintessence [10] would moreover allow emergence, after recombination, of an attractor solution which comfortably solves the problem of cosmic coincidence [11] and may improve, though does not solve, that of the cosmological constant. In spite of all the successes that modern quintessence has already achieved, as it happened with its ancient Greek ancestor, it retains the shortcoming that one can wonder, where would such a field come from?. A recent idea that may conveniently answer this question is to consider [12] that the quintessence field springs from the physics of extra dimensions, or in other words, that this field should be a natural component of the higher dimensional bulk of the brane worlds, and was created when the universe was created from nothing or “something”. It then appears an appealing idea to consider the instantonic creation of a brane world whose five-dimensional bulk contains a negative cosmological constant and a scalar quintessence field. This paper aims at implementing this idea within the framework of five-dimensional Euclidean gravity, restricting ourselves to the case that the quintessence field minimally couples to Hilbert-Einstein gravity.

The physics of the brane worlds created in a bulk with extra dimensions is other fundamental issue [13] in present cosmology that may help to solve the long standing hierarchy problem [14]. Our approach in this paper involves an instantonic procedure [15] which, without resorting to any particular particle-physics models, will allow us to deal with the process of creation of a brane world with a quintessence field, using only the machinery of canonical semiclassical and quantum cosmology. As to the boundary conditions for a universe created this way, we shall adopt the general notion of creation from nothing [16], but only in the sense of allowing also the possibil-

ity that the universe was created from a baby universe that evolved as a component of the quantum spacetime foam, and actually take these two creative mechanisms (which e.g. respectively entail a no boundary initial condition [17] or a regular wave functional as the geometry degenerates [18]) to be simultaneously contributing the quantum state of the universe. It can be noted that such a generalized boundary condition is indeed compatible with current ideas on the early evolution of a tracking quintessence field in four dimensions. Thus, as one goes back in cosmological time passing through the distinct (potential, transition and kinetic) quintessential regimes, each characterized by given nearly constant values of ω [19], one finally arrives at very high redshifts of the order $z \geq 10^{30}$ where, according to the classical equations, ω starts decreasing from $\omega = +1$. However, at those high values of z , the universe had already actually entered a quantum regime where all possible values at least from $+1$ to -1 are allowed for ω .

The paper can be outlined as follows. In Sec. II we describe the general Euclidean formalism for a homogeneous scalar quintessence field which is minimally coupled to Hilbert-Einstein gravity in the presence of a negative cosmological constant, both in the five- and four-dimensional cases. From the equations that define the quintessence field and its conservation law, we finally obtain a Hamiltonian constraint equation which is expressed in terms of the parameter ω . Sec. III deals with the classical solution and the quantum state that correspond to the particular state equation $\omega = -3/4$. It is shown that the classical solution describes a new asymptotically anti-de Sitter (AdS) Euclidean wormhole and that, if we cut the manifold at the wormhole throat, the wave function describes a pure quantum state which can be given in terms of parabolic cylinder functions. We also consider that the quantum state should be given as a mixed density matrix when the manifold cannot be cutted in two disconnected parts and the instantonic processes leading to either a single brane or a string of brane-antibrane pairs, and their respective quantum states. Other particular classical and quantum Euclidean solutions corresponding to different values of ω covering the entire range of its permissible values are analysed in Sec. IV, where the role of such solutions in the construction of brane worlds is also discussed in some detail. Sec. V contains a general prescription for the cosmological evolution of the inflating branes after their creation. A tentative model is also constructed for the evolution of each brane along a cosmological time in which, starting from the five-dimensional scenario, one arrives at a general expression relating the scale factor of the observable universe with the quintessence state equation and the metrics analysed above on the brane hypersurface. Finally, we summarize and add some comments on the nucleation probability for generalized solutions and the graviton ground state and spectrum of the Kaluza-Klein modes in Sec. VI.

II. QUINTESSENCE DYNAMICS IN ADS SPACE

In this section we will consider first the general-relativistic dynamics of a homogeneous quintessential scalar field, ϕ , which is minimally coupled to Hilbert-Einstein gravity, in a five-dimensional framework, in the presence of a negative cosmological constant Λ_5 . The inclusion of this negative cosmological constant supports recent claims [20] that AdS or asymptotically AdS spaces had to have played a rather decisive role in primordial particle and cosmological physics, and offers at the same time the opportunity to construct consistent brane-world scenarios to deal with cosmological and gravity problems. Originally, quintessence fields were considered [8] to be slowly varying in order for the pressure entering their equation of state to be always negative. However, in more recent quintessence-tracking models [10,19] where that equation of state is not restricted to be constant, the quintessence fields are not necessarily assumed to be slowly varying along the entire cosmological range of redshift values, but to satisfy a time-variable state equation with the general form

$$p = \omega(t)\rho, \quad (2.1)$$

where p and ρ are, respectively, the pressure and energy density for the quintessence field, and $\omega(t)$ is the time-dependent parameter specifying the equation, with

$$+1 < \omega < -1. \quad (2.2)$$

As it was pointed out in the Introduction, the time evolution of parameter ω in quintessence tracking models is not specified as one approaches the semiclassical and quantum regions that correspond to redshifts larger than $\sim z = 10^{30}$, so that one should consider all possible values of ω , restricted only by Eqn. (2.2), as such regions are considered. We therefore study a general-relativistic model for an unspecified equation of state for the quintessence field in the presence of a negative cosmological constant. Thus, in order for dealing with the semiclassical and quantum states of our whole system, instead of choosing the solution for a particular value of ω , we shall consider a set of such solutions and interpret them as describing contributing semiclassical or quantum paths.

The Euclidean action for the five-dimensional system formed by a homogeneous quintessential scalar field ϕ with unspecified state equation and potential $V(\phi)$, which is minimally coupled to Hilbert-Einstein gravity in the presence of a negative cosmological constant Λ_5 , can be written as

$$I = -\frac{1}{16\pi G_5} \int d^5x \sqrt{g} (R + 2\Lambda_5) \\ + \frac{1}{2} \int d^5x \sqrt{g} \left[(\nabla\phi)^2 + 2V(\phi) \right]$$

$$-\frac{1}{8\pi G_5} \int d^4x \sqrt{h} \text{Tr} K, \quad (2.3)$$

where G_5 is the five-dimensional gravitational constant, g and h are the determinant of the five-metric and the metric induced on the boundary four-surface, respectively, R is the scalar curvature, and K is the second fundamental form on the boundary surface. Assuming an ansatz for the Euclidean metric with topology $\mathbb{R} \times \mathbb{S}^4$, i.e.

$$ds^2 = N^2 d\tau^2 + a(\tau)^2 d\Omega_4^2, \quad (2.4)$$

in which τ is the extra direction and N is the lapse function, with $d\Omega_4^2$ the metric on the unit four-sphere and the actual time coordinate for observers in \mathbb{S}^4 being one of the coordinates of that sphere (see Sec. V), the action (2.3) reduces to

$$I = -\frac{3}{4\pi G_5} \int d^5x N a^2 \times \left[\frac{\dot{a}^2}{N^2} + 1 + \Lambda a^2 + \frac{2\pi G_5}{3} \left(\frac{\dot{\phi}^2}{N^2} + 2V(\phi) \right) a^2 \right], \quad (2.5)$$

where the overhead dot denotes τ -derivative and $\Lambda = -\Lambda_5/6$. We thus note that even though τ is not a true time coordinate for observers in \mathbb{S}^4 , relative to the unobservable five-dimensional space as a whole, it plays the role of an Euclidean time.

From action (2.5) we can obtain the field equations for the scale factor a and the field ϕ and the Hamiltonian constraint. In the gauge $N = 1$, they are:

$$\frac{d}{d\tau} (a\dot{a}) = 1 + \Lambda a^2 + \frac{4\pi G_5 a^2}{3} [\dot{\phi}^2 + 2V(\phi)] \quad (2.6)$$

$$\ddot{\phi} + 4\dot{\phi} \frac{\dot{a}}{a} = \frac{dV(\phi)}{d\phi} \quad (2.7)$$

$$-\dot{a}^2 + 1 + \Lambda a^2 - \frac{2\pi G_5}{3} (-\dot{\phi}^2 + 2V(\phi)) a^2 = 0. \quad (2.8)$$

The above expressions can be simplified if we now use the definition of the five-dimensional quintessence field ϕ in terms of the pressure p and the energy density ρ . For that definition we will simply take the extension of the usual four-dimensional definition of the quintessence field [8] to the five-dimensional case, choosing the extra coordinate τ to play the role of an Euclidean time. If we rotate this to its "Lorentzian" counterpart, $\tau \rightarrow -it$, then we have the following definition for the field ϕ in five dimensions:

$$8\pi G_5 \rho = \frac{1}{2} \dot{\phi}^2 + V(\phi) \quad (2.9)$$

$$8\pi G_5 p = \frac{1}{2} \dot{\phi}^2 - V(\phi), \quad (2.10)$$

where the overhead dot means now d/dt . As referred to a four-sphere, we can also introduce a conservation law for field ϕ given by

$$\rho = \rho_0 a^{-4(1+\omega)}, \quad (2.11)$$

in which ρ_0 is a constant. In particular, using Eqns. (2.9)-(2.11), we obtain a relation between the Euclidean counterparts of the energy density for the quintessence field and the scale factor given by

$$-\dot{\phi}^2 + 2V(\phi) = 16\pi G_5 \rho_0 a^{-4(1+\omega)}, \quad (2.12)$$

with which the Hamiltonian constraint (3.8) reduces to

$$\dot{a}^2 - \Lambda a^2 + A^2 a^{-2(1+2\omega)} = 1, \quad (2.13)$$

where $A^2 = 32\pi^2 G_5^2 \rho_0 / 3$ is a constant. From the state equation $p = \omega \rho$ and the definition of ϕ given by Eqns. (2.9) and (2.10) it follows that, if we want to preserve an interval $+1 > \omega > -1$ for ω also in the Euclidean region, then the t -dependence of ϕ and the shape of potential $V(\phi)$ should be such that the rotation $t \rightarrow i\tau$ must necessarily entail a change of sign of potential $V[\phi(t)] \rightarrow -V[\phi(\tau)]$ which satisfies the equations of motion.

Although we have not been able to find any closed-form solutions to Eqn. (2.13) for a generic ω , some analytical solutions for particular values of ω will be obtained and discussed in Secs. III and IV. In what follows of the present section, we shall restrict ourselves to briefly consider the case in which the coupling between gravity and the quintessence field takes place in the presence of a negative cosmological constant in four dimensions. One can readily show that in that case the Euclidean action is like Eqn. (2.3), but for a Newton gravitational constant G_N instead G_5 , with the integration being now over four dimensions, i.e. over d^4x in the main terms and over d^3x in the surface term. After applying the general-relativity machinery for a minisuperspace $ds^2 = d\tau^2 + a(\tau)^2 d\Omega_3^2$ (with $d\Omega_3^2$ the metric on the three-sphere) we obtain then

$$\dot{a}^2 - 1 - \Lambda a^2 + \frac{2\pi G_N}{3} [-\dot{\phi}^2 + 2V(\phi)] a^2 = 0. \quad (2.14)$$

Using the definition of the four-dimensional quintessence field [8], which is also given by Eqns. (2.9) and (2.10), and a conservation law referred to a three-surface for it [21], $\rho = \rho_{40} a^{-3(1+\omega_4)}$, we finally obtain for the Hamiltonian constraint

$$\dot{a}^2 - \Lambda a^2 + A_4^2 a^{-(1+3\omega_4)} = 1, \quad (2.15)$$

where the constant in the third term is now given by $A_4^2 = 32\pi^2 G_N^2 \rho_{40} / 3$. Comparison of expressions (2.13) and (2.15) leads to the conclusion that all possible solutions to the constraint (2.15) for a particular equation of state ω_4 in four dimensions should be the same as the solutions to Eqn. (2.13) for an equation of state specified by a parameter $\omega = 3\omega_4/4 - 1/4$ in five dimensions. In the two following sections we shall explicitly consider some of such solutions.

III. THE $\omega = -3/4$ FIVE-DIMENSIONAL WORMHOLE

If we set the equation of state for the quintessential field to be $\omega = -3/4$, then from Eqn. (2.13) we obtain a Lorentzian Hamiltonian constraint given by

$$-\dot{a}^2 - \Lambda a^2 + A^2 a = 1. \quad (3.1)$$

This constraint is equivalent to the equation that describes the motion of a particle in a potential

$$U(a) = \frac{1}{2} (\Lambda a^2 - A^2 a + 1),$$

with zero total energy. Potential $U(a)$ has its minimum at $a_{\min} = A^4/(8\Lambda)$. From the constraint equation (3.1), one can see that an Euclidean solution for time $\tau \rightarrow -it$ can only exist when $\Lambda > A^4/4$. Solutions to Eqn. (3.1) which are nonsingular and periodic (in Lorentzian time) lie in the region $a_- < a < a_+$, with

$$a_{\pm} = \frac{A^2 \pm \sqrt{A^4 - 4\Lambda}}{2\Lambda}. \quad (3.2)$$

The Euclidean solution will be then

$$a(\tau) = \frac{A^2 \pm \sqrt{\alpha} \sinh(\sqrt{\Lambda}\tau)}{2\Lambda}, \quad (3.3)$$

where $\alpha = 4\Lambda - A^4 > 0$, and the upper + sign accounts for the region with $\tau > 0$ and the lower - sign accounts for the region with $\tau < 0$. This is a new asymptotically anti-de Sitter Euclidean wormhole solution, which is qualitatively the same, but formally simpler than the one that corresponds to a five-dimensional $\omega = 0$ state equation [22,27] (see Sec. IV). As pointed out at the end of Sec. II, a solution which is formally the same as that is given by Eqn. (3.3) can be obtained also in the four-dimensional case for an equation of state $\omega_4 = -2/3$, i.e. in the quintessential regime which is able to reproduce an accelerating universe.

Four-dimensional Euclidean wormholes have always been looked at with some mistrust because they evolve in Euclidean time and therefore somehow "live in eternity". Perhaps the physics of extra dimensions may provide a suitable framework for wormholes to become more acceptable physically. In fact, five-dimensional wormhole solutions, such as the one given by Eqn. (3.3), would evolve along the fifth direction of either a positive definite metric, or a metric with Lorentzian signature having as time direction one of the four-sphere (see Sec. V). In the latter case, wormholes would no longer show the above shortcoming.

In the Lorentzian region, we obtain a real solution only for the case $A^4 > 4\Lambda$, that is

$$a(t) = \frac{A^2 + \sqrt{-\alpha} \sin(\sqrt{\Lambda}t)}{2\Lambda},$$

which varies from the radius of the wormhole throat $a_0 = A^2/(2\Lambda)$, at $t = 0$, first up to a maximum radius $(A^2 + \sqrt{-\alpha})/(2\Lambda)$, then down to $(A^2 - \sqrt{-\alpha})/(2\Lambda)$, to finally return to a_0 again, as t completes its period at $t = 2\pi/\sqrt{\Lambda}$. This Lorentzian solution gives the scale factor of a closed nonsingular baby universe.

Moreover, one can express solution (3.3) in term of the Euclidean conformal time η defined by

$$\eta = \int \frac{d\tau}{a(\tau)} =$$

$$\eta_{\pm}^{\pm} \pm 2\text{arctanh} \left[\frac{A^2 \tanh(\sqrt{\Lambda}\tau/2) - \sqrt{\alpha}}{2\sqrt{\Lambda}} \right], \quad (3.4)$$

where

$$\eta_{\pm}^{\pm} = \pm 2\text{arctanh} \left(\frac{\sqrt{\alpha}}{2\sqrt{\Lambda}} \right), \quad (3.5)$$

with the upper sign + in the above equations accounting for the conformal time interval $0 \leq \eta \leq \eta_{+}^{+} + 2\text{arctanh} \left[(A^2 - \sqrt{\alpha}) / (2\sqrt{\Lambda}) \right]$, and the lower sign - accounting for $0 \leq \eta \leq \eta_{-}^{-} - 2\text{arctanh} \left[(A^2 - \sqrt{\alpha}) / (2\sqrt{\Lambda}) \right]$. In this way, solution (3.3) can be re-written

$$a(\eta) = \frac{2A^2}{\alpha \cosh \eta - 2\sqrt{\alpha\Lambda} \sinh \eta + A^4}. \quad (3.6)$$

We consider next the quantum state of the above Euclidean wormhole under the following three usual assumptions: (1) we disregard all factor ordering problem related with the momentum operator, (2) the quantum state is restricted to be a pure state describable by means of a wave function, and (3) we obtain the wave functional by using the Wheeler de Witt formalism. Assumption (2) implies that the wormhole manifold is simply connected in the sense that one can cut it at the wormhole throat $a_0 = A^2/(2\Lambda)$ to get two disconnected half-wormhole submanifolds. If the cutting would not divide the manifold into two disconnected parts, then the quantum state of the wormhole would be given by a mixed density matrix [23]. One can also obtain the quantum state by using the Euclidean path-integral formalism. However, we shall restrict ourselves here to derive the expression for the wave function by using the equivalent formalism of the Wheeler de Witt equation. This is constructed starting with the classical Hamiltonian constraint (3.1) by replacing the momentum $\pi_a = \dot{a}$, conjugate to the scale factor a , for the quantum-mechanical operator $1/(2\sqrt{\Lambda})(\delta/\delta a)$. If we disregard the factor ordering problem, then we can choose for the wave equation

$$\left[-\frac{1}{4\Lambda} \frac{\partial^2}{\partial a^2} + (\Lambda a^2 - A^2 a + 1) \right] \Psi(a) = 0, \quad (3.7)$$

where $\Psi(a)$ is the wave function for each half-wormhole. Introducing the change of variable

$$z = 2\sqrt{\Lambda}a - \frac{A^2}{\sqrt{\Lambda}},$$

where the new coordinate z varies from 0, at the wormhole neck, up to ∞ in the asymptotic region, the above wave equation can be recast in the form

$$\left(-\frac{\partial^2}{\partial z^2} + \frac{1}{4}z^2 + 1 - \frac{A^4}{4\Lambda}\right) \Psi(z) = 0. \quad (3.8)$$

Eqn. (3.8) is a differential equation defining a parabolic cylinder function D [24,25]. We have then the set of linearly dependent wave functions characterizing the pure quantum state of the considered Euclidean asymptotically AdS wormhole:

$$\Psi_1^\pm = D_{\frac{A^4}{4\Lambda} - \frac{3}{2}} \left[\pm \left(2\sqrt{\Lambda}a - \frac{A^2}{\sqrt{\Lambda}} \right) \right] \quad (3.9)$$

$$\Psi_2^\pm = D_{\frac{1}{2} - \frac{A^4}{4\Lambda}} \left[\pm i \left(2\sqrt{\Lambda}a - \frac{A^2}{\sqrt{\Lambda}} \right) \right]. \quad (3.10)$$

Since $\lim_{a \rightarrow \infty} \Psi \rightarrow 0$ and $\lim_{a \rightarrow 0} \Psi \rightarrow 0$ for all of these wave functions, they must satisfy the Page-Hawking boundary conditions for the wave function of Euclidean wormholes [18].

In order to construct brane worlds starting with a five-dimensional Euclidean metric

$$ds^2 = d\tau^2 + a(\tau)^2 d\Omega_4^2, \quad (3.11)$$

where the scale factor $a(\tau)$ is given by Eqn. (3), we shall follow the procedure put forwards by Garriga and Sasaki for the five-dimensional AdS instanton [15]. In case that the manifold be divisible in disconnected parts by cutting along the radial extra coordinate, two different situations can be distinguished. On the one hand, the wormhole manifold is cutted at the neck at $\tau = 0$ and at a given hypersurface $\tau = \tau_0$ (or $\tau = -\tau_0$), excising the remaining spacetime $\tau < 0$ and $\tau > \tau_0$ (or $\tau > 0$ and $\tau < -\tau_0$). Gluing then the resulting space with a copy of it at the hypersurface $\tau = \tau_0$ (or $\tau = -\tau_0$), we obtain a single brane if we introduce a brane tension given by [26]

$$\sigma_\pm = \pm \frac{3\sqrt{\alpha\Lambda} \cosh\left(\sqrt{\Lambda}\tau_0\right)}{4\pi G_5 \left(A^2 \pm \sqrt{\alpha} \sinh\left(\sqrt{\Lambda}\tau_0\right)\right)}, \quad (3.12)$$

where the upper/lower sign stands for $\tau > / < 0$. On the other hand, if the entire wormhole manifold is cutted at a given $\tau_0 > 0$ and at a given $\tau'_0 < 0$, but not at $\tau = 0$, excising the remaining spacetime $\tau > \tau_0$ and $\tau < \tau'_0$, then one can glue a succession of copies of the resulting spacetime with the τ -orientation reverse of the respective copies to produce an instanton consisting of an infinite

(or finite if we cut at some $\tau = 0$ at right and left) string of brane-antibrane pairs, each brane with a tension given by Eqn. (3.12) with the upper sign and each antibrane with the same tension but with the lower sign. We note that the brane's tension given in all the above cases by Eq. (3.12) becomes $\pm 3\sqrt{\alpha\Lambda}/(4\pi G_5 A^2)$ in the limit $\tau_0 \rightarrow 0$, and approaches the critical Randall-Sundrum value [14] $\sigma_c = 3\sqrt{\Lambda}/(4\pi G_5)$ as $\tau_0 \rightarrow \infty$. In the limit $A^2 \rightarrow 0$, the tension (3.12) reduces to the singular expression that corresponds to the case of a pure five-dimensional AdS space, $3\sqrt{\Lambda} \coth\left(\sqrt{\Lambda}\tau_0\right)/(4\pi G_5)$, such as it happens in the case of the asymptotically AdS wormhole for $\omega = 0$ studied in Ref. [27].

In the first of the above situations, the resulting single-brane instanton can be illustrated as two opposite Euclidean cups, each with a Lorentzian baby universe replacing the south pole at Euclidean time $\tau = 0$, which interset each other at $\tau = \tau_0$. This corresponds to two copies of the spherical patch of the $\omega = -3/4$ asymptotically AdS half-wormhole bulk which are bounded by a common four-sphere. Such a four-sphere is the world-sheet of the single brane. In this way, the single-brane instanton can be interpreted as a semiclassical path for the creation of a brane world (containing a bulk $\omega = -3/4$ Euclidean asymptotically AdS wormhole) from a baby universe and, when it is cutted in half, the solution interpolates between the baby universe that replaces the south pole at $\tau = 0$, and a spherical brane of radius $H^{-1} = a(\tau_0)$ at the equator. As it was pointed out by Garriga and Sasaki [15] and Bouhmadi and González-Díaz [27], a construction like this is analogous to the four-dimensional de Sitter instanton [28], though the present instanton describes creation of a universe from a baby universe, rather than nothing. In the model describing creation of a string of brane-antibrane pairs, the whole instanton could be illustrated as an indefinite chain whose beads are made of two opposite cups smoothly matching each other at $\tau = 0$ (the throat) and interset the two neighbouring beads at, respectively, $\tau = \tau_0 > 0$ and $\tau = \tau'_0 < 0$ (see Fig. 1). The bounding common four-spheres at $\tau_0 > 0$ are the world-sheet for branes and the bounding common four-spheres at $\tau'_0 < 0$ are the world-sheet for antibranes, and the whole instanton is to be regarded as a semiclassical path for the creation of a string of brane-antibrane worlds from a string of baby universes.

In the case that the four-surface S on which we take the data does not divide the manifold into two disconnected parts, we have to use a mixed state density matrix $\rho(a)$ [23], rather than a wave function $\Psi(a)$. This is usually computed by using a propagator $K[h_{ij}, 0; h'_{ij}, \tau_1]$ for the wave function from the data $[h_{ij}]$ (four-metric) on a four-surface S at $\tau = 0$ to the data $[h'_{ij}]$ on another four-surface S' at $\tau = \tau_1$. This propagator can be written as

$$K[h_{ij}, 0; h'_{ij}, \tau_1] = \sum_n \Psi_n[h_{ij}] \Psi_n[h'_{ij}] e^{-n\sqrt{\Lambda}\tau_1}, \quad (3.13)$$

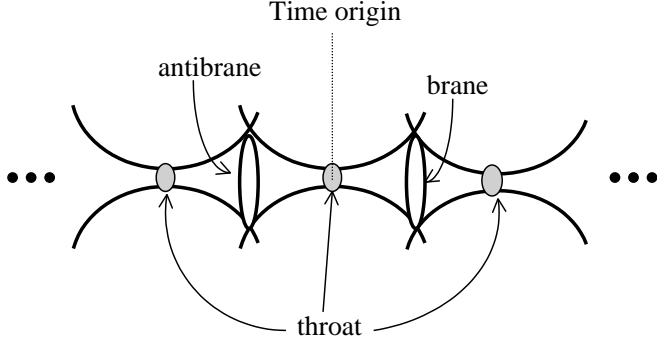


FIG. 1. Instanton for the creation of a string of brane-antibrane worlds. Vertical circles represent four-sphere branes or antibranes at which two $\omega = -3/4$ asymptotically AdS wormholes are glued introducing a positive or negative tension, respectively.

where Ψ_n represents the state for the wormhole with an assumed discrete spectrum labelled by the index n . The quantum state given by a parabolic cylinder function shows nevertheless a continuous spectrum. However, one could still single out a discrete spectrum for the wormhole by assuming values for the parameters Λ and A^2 such that the index of the parabolic cylinder function $A^4/4\Lambda - 3/4$ in the wave function (3.10) takes only on positive integer values. Under such an assumption, we can re-express the wave function in terms of Hermite polynomials [24], so that for the upper sign in solution (3.10) we have

$$\Psi_n(n) \propto$$

$$e^{-\frac{1}{2}(2\sqrt{\Lambda}a - A^2/\sqrt{\Lambda})^2} H_{A^4/4\Lambda - 3/2} \left(\sqrt{2\Lambda}a - \frac{A^2}{\sqrt{2\Lambda}} \right). \quad (3.14)$$

However, if we insist in having positive definite values for the discrete index, then this wave function cannot be taken to represent the quantum state in the Euclidean regime, as that regime is characterized by $4\Lambda > A^4$ for which index n is negative definite. For the Lorentzian regime where $4\Lambda < A^4$ and the index is positive definite the contribution to the density matrix from five-geometries that the four-surface S does not divide will be given by the propagator

$$K[a, 0; a', t_1] = \sum_n \Psi_n[a] \Psi_n[a'] e^{-i(n+1/2)\sqrt{\Lambda}t_1}. \quad (3.15)$$

Because S and S' may have any Lorentzian time separation t_1 in order to obtain the density matrix one has to integrate over all values of t_1 from 0 to $\pi/\sqrt{\Lambda}$. This gives

$$\begin{aligned} \rho[a; a'] &= i \text{Re} \int_0^{\pi/\sqrt{\Lambda}} dt_1 K[a, 0; a', t_1] \\ &= 2 \sum_{n=0}^{\infty} \frac{\Psi_n[a] \Psi_n[a']}{(n+1/2)\sqrt{\Lambda}}, \end{aligned} \quad (3.16)$$

where $n = A^4/(4\Lambda) - 3/2$. Eqn. (3.16) can be interpreted by considering that the baby universes associated with the wormhole are in the quantum state specified by the wave function $\Psi_n[a]$, with a relative probability given by

$$P_n = \frac{2}{(n+1/2)\sqrt{\Lambda}}.$$

Since $A^4 > 4\Lambda$ in the Lorentzian region, the density matrix given by Eqn. (3.16) is positive definite and can never diverge.

Finally, the wave function that describes the quantum state of a single instantonic brane world with the brane at an arbitrary $a = a_0$ may also be obtained from the single half-wormhole quantum states given by Eqns. (3.9) and (3.10). Because all of the spacetime corresponding to $a > a_0$ should here be excised off, in order to construct the quantum state of a single brane world, one should integrate the wave functions for the complete half-wormhole over all values of the scale factor a from the brane position at a_0 to the asymptotic region where $a \rightarrow \infty$. If we generically represent the wave function given by expression (3.9) by $\Psi_r = D_r(z)$, with z as given in Eqn. (3.8) and $r = A^4/(4\Lambda) - 3/2$, we will then have for the pure quantum state of a single brane world

$$\begin{aligned} \Psi_r^{(b)}(z) &= \int_{z_0}^{\infty} dz D_r(z) \\ &= \frac{1}{2} \int_{z_0}^{\infty} dz z D_{r+1}(z) + D_{r+1}(z)|_{z_0}^{\infty}. \end{aligned} \quad (3.17)$$

States like this are associated with a continuous spectrum labelled by the index $r+1$ which is bounded from above at $r = -1$ due to the Euclidean restriction $4\Lambda > A^4$. However, if we would assume r to take only on integer values and then re-express the state (3.17) in terms of Hermite polynomials [24], it can be readily seen that for $r = -1$ the quantum state becomes $\Psi_{-1}^{(b)} \propto \exp(-z_0^2/4)$. Thus, the spectrum associated with the quantum states of a single brane world constructed starting with a five-dimensional $\omega = -3/4$ wormhole is generally continuous and runs along the index r which takes on continuous negative values which are bounded from above by $r \leq -1$.

IV. OTHER ADS OR ASYMPTOTICALLY ADS SOLUTIONS

In this section we shall consider other possible solutions to the constraint equation (2.13) and discuss their role in the construction of brane-world instantons similar to those dealt with in the precedent section. We will study the cases arising from a quintessential field which is minimally coupled to Hilbert-Einstein gravity in five and four dimensions. Let us start with the five-dimensional

example for constant equations of state and the general Euclidean Hamiltonian constraint

$$\dot{a}^2 - \Lambda a^2 + A^2 a^{-(4\omega+2)} = 1, \quad (4.1)$$

which corresponds to an instantonic metric

$$ds^2 = d\tau^2 + a(\tau)^2 d\Omega_4^2. \quad (4.2)$$

We shall regard the cases corresponding to particular values of the parameter ω within the interval $-1 < \omega < +1$. Thus, for $\omega = +1/4$, we obtain

$$\dot{a}^2 - \Lambda a^2 + A^2 a^{-3} = 1. \quad (4.3)$$

The Hamiltonian constraint (4.3) also corresponds to the case of a massless, homogeneous non-quintessential scalar field Φ , conformally coupled to Hilbert-Einstein gravity in five dimensions, for which the Euclidean action reads

$$I = -\frac{1}{16\pi G_5} \int d^5x \sqrt{g} \left[\left(1 - \frac{3\pi G_5 \Phi^2}{2} \right) R + 2\Lambda_5 \right] \\ + \frac{1}{2} \int d^5x \sqrt{g} (\nabla \Phi)^2 \quad (4.4)$$

$$-\frac{1}{8\pi G_5} \int d^4x \sqrt{h} \left(1 - \frac{3\pi G_5 \Phi^2}{2} \right) \text{Tr} K,$$

where K is the extrinsic curvature on the chosen four-boundary. For an Euclidean metric of the form

$$ds^2 = N^2 d\tau^2 + a^2(\tau) d\Omega_4^2 = a(\eta)^2 (N^2 d\eta^2 + d\Omega_4^2), \quad (4.5)$$

where N again is the lapse function and $\eta = \int d\tau/a$ the conformal time, the Euclidean action becomes

$$I = -\frac{3}{4\pi G_5} \int d^5x N \left(a^2 \frac{\dot{a}^2}{N^2} + a^2 + \Lambda a^4 + \frac{\chi^2}{a} + \frac{4}{9} a \frac{\dot{\chi}^2}{N^2} \right), \quad (4.6)$$

in which we have redefined $\Lambda = -\Lambda_5/6$ and $\chi = a^{3/2} \sqrt{3\pi G_5/2} \Phi$. In the gauge $N = 1$, we can derive from this Euclidean action the Hamiltonian constraint and the equation of motion for the field χ . In terms of the conformal time η these quantities are given by

$$-a'^2 + a^2 + \Lambda a^4 + \frac{1}{a} \left(\chi^2 - \frac{4}{9} \chi'^2 \right) = 0 \quad (4.7)$$

$$\chi'' = \frac{9}{4} \chi. \quad (4.8)$$

A rather trivial solution to Eqn. (4.8) is

$$\chi = A \sinh\left(\frac{3}{2}\right) + B \cosh\left(\frac{3}{2}\right), \quad (4.9)$$

where A and B are arbitrary constants. Hence, we obtain for the equation of motion for the scale factor,

$$aa'' + a'^2 - a^2 - 2\Lambda a^4 + aR_0^2 = 0, \quad (4.10)$$

in which R_0^2 is an integration constant, and for the Hamiltonian constraint the same expression as for the case $\omega = +1/4$ given by Eqn. (4.3). We have not been able to find a solution to Eqn. (4.3) in closed form, but only approximate expressions in limiting cases. Thus, for small a , we obtain $a \simeq A^{2/3} \cosh^{2/3}(3\eta/2)$, and for large a , the pure AdS solution. This suggests that we have an asymptotically AdS Euclidean wormhole again for the minimally coupled quintessence field with state equation $\omega = +1/4$ in five dimensions. This solution also corresponds to the equation of state $\omega_4 = +2/3$ in four dimensions. Therefore, for the former case, the cutting and pasting procedure explicited for the equation of state $\omega = -3/4$ would allow us to construct essentially the same types of brane-world instantons also in this case, that is, either a single inflating brane world, or a string of brane-antibrane worlds. The quantum states for these wormholes and derived brane worlds will be studied elsewhere.

If we fix the equation of state to describe a pressureless quintessence field, $\omega = 0$, then we obtain from Eqn. (4.1)

$$\dot{a}^2 - \Lambda a^2 + \frac{A^2}{a^2} = 1. \quad (4.11)$$

The same expression but referred to the constant A_4 is also obtained in the four-dimensional case for a quintessence-field state equation $\omega_4 = +1/3$. It corresponds, on the other hand, to the case of a homogeneous, massless non-quintessential scalar field conformally coupled to Hilbert-Einstein gravity in four dimensions. In terms of the coordinate τ , the solution of Eqn. (4.11) can be given in closed form and reads:

$$a(\tau) = \left[\frac{\sqrt{\beta} \cosh(2\sqrt{\Lambda}\tau) - 1}{2\Lambda} \right]^{1/2}, \quad (4.12)$$

where

$$\beta = 1 + 4\Lambda A^2. \quad (4.13)$$

As expressed in terms of the conformal coordinate $\eta = \int d\tau/a$, solution (4.12) becomes

$$a(\eta) = \sqrt{\frac{\sqrt{\beta}-1}{2\Lambda}} \text{nc}(\beta^{1/4}\eta), \quad (4.14)$$

with nc a Jacobian elliptic function [24]. This solution represents an asymptotically AdS Euclidean wormhole which was first studied by González-Díaz [29] and

Barceló et al. [22], both in the classical and quantum four-dimensional cases, which are here associated with an equation of state $\omega_4 = +1/3$. Nevertheless, even though the classical solution (4.12) is formally the same as that was derived by conformally coupling a massless scalar field to Euclidean four-dimensional Hilbert-Einstein gravity [30], the quantum mechanical treatment given in Ref. [22] no longer applies to the present case because here the scalar field couples minimally to five-dimensional gravity and is subjected to the specific definitory characteristics of a quintessence field constraining it to depend on a according to Eqn. (2.12). Thus, starting with the Wheeler de Witt equation derived from Eqn. (4.11) by following the same procedure as in Sec. III, and redefining the scale factor so that $x = \sqrt{2\Lambda}a$, we get

$$\left[\frac{\partial^2}{\partial x^2} - \left(x^2 - \frac{4\Lambda A^2}{x^2} + 2 \right) \right] \Psi(x) = 0. \quad (4.15)$$

Therefore, instead of a function depending on a discrete index such as the Hermite polynomials, solutions to the wave equation can be generally given in terms of the confluent hypergeometric functions and read

$$\Psi_{\pm}(a) = \left(\sqrt{2\Lambda}a \right)^{(1 \pm B)/2} e^{-\Lambda a^2} \Theta \left(1 \pm \frac{1}{4}B, 1 \pm \frac{1}{2}B, 2\Lambda a^2 \right), \quad (4.16)$$

where $B = \sqrt{1 - 16\Lambda A^2}$ and Θ represents the Kummer's functions M and U [24]. The pure quantum states given by solutions (4.16) are regular as the geometry degenerates at $a = 0$ and exponentially damp as $a \rightarrow \infty$, so providing satisfactory quantum states to represent a wormhole. These states will generally span a continuous spectrum as one lets the parameters λ and A^2 to continuously vary. For the limiting case in which $B = 0$, the quantum state would reduce to the wave functions (4.16) with the confluent hypergeometric function Θ replaced for the generalized Laguerre polynomial $L_{-1}^{(0)}(2\Lambda a^2) \equiv L_{-1}(2\Lambda a^2)$, which identically vanishes.

The use of the instanton (4.12) in the five-dimensional case for the construction of inflating brane worlds has recently been made by Bouhmadi and González-Díaz [27]. It leads essentially to the same qualitative picture as that we have described for the wormhole obtained for a state equation $\omega = -3/4$ in the five-dimensional case (or $\omega_4 = -2/3$ in four dimensions). Thus, the quantum state of such brane worlds would be obtained by integrating the quantum states (4.16) over a from a_0 to $a \rightarrow \infty$. Also in this case there is no quantum state for the limiting value $B = 0$,

$$\Psi^{(b)} = \int da \left(\sqrt{2\Lambda}a \right)^{1/2} e^{-\Lambda a^2} L_{-1}(2\Lambda a^2) = 0,$$

so that there could be no brane world being created in such a limiting case.

We consider next the situation created in the five-dimensional framework when $\omega = -1/4$ (or $\omega_4 = 0$ in four dimensions). Eqn. (4.1) becomes then

$$\dot{a}^2 - \Lambda a^2 + \frac{A^2}{a} = 1. \quad (4.17)$$

We have not obtained solutions to this equation in closed form, but only approximate expressions for the limiting cases. In fact, whereas for small a we have the wormhole-neck behaviour

$$a \simeq \frac{A^2}{2} (1 + \sinh \eta) \simeq \frac{A^2}{2} \sqrt{1 + \frac{4\tau}{A^2}}, \quad (4.18)$$

an asymptotic pure AdS behaviour is obtained at large a . That is, although its spacetime structure is quite more complicated than the solution dealt with in Sec. III, one would expect the solution to Eqn. (4.17) to also describe an Euclidean asymptotically AdS wormhole, and to lead to brane-world instantons having essentially the same characteristics as those derived for the $\omega = -3/4$ five-dimensional wormhole. Explicit expressions for the possible quantum states of these wormholes and brane worlds will be also considered elsewhere.

Finally, let us briefly look at the five-dimensional case $\omega = -1/2$ (or $\omega_4 = -1/3$ in four dimensions). We have in this case

$$\dot{a}^2 - \Lambda a^2 + A^2 = 1. \quad (4.19)$$

The solution to Eqn. (4.19) is (for $A^2 < 1$):

$$a = \sqrt{\frac{1 - A^2}{\Lambda}} \sinh(\sqrt{\Lambda}\tau) = \frac{\sqrt{1 - A^2}}{\sqrt{\Lambda} \sinh(\sqrt{1 - A^2}\eta)}, \quad (4.20)$$

that is, pure AdS space with radius $r(A) \equiv \sqrt{\frac{1 - A^2}{\Lambda}}$. The construction of brane worlds from Euclidean five-dimensional AdS instanton was considered by Garriga and Sasaki for the extreme case at $A^2 \rightarrow 0$ [15], i.e. for a AdS radius $r(0) \equiv 1/\sqrt{\Lambda}$. Hence, after conveniently replacing $r(0)$ for $r(A)$, the construction of brane-world instantons here will follow exactly the same pattern as what was made by these authors. We thus obtain single brane instantons, but no string of brane-antibrane instantons can be constructed. On the other hand, the generalized solution that follows to (4.2) in this case is singular in the sense that the Weyl tensor diverges as $\tau \rightarrow 0$, a shortcoming which is not present in all the generalized solutions that correspond to the asymptotically AdS wormhole instantons considered in this work.

Let us consider the quantum state that can be defined for our AdS space. By applying the usual correspondence principle to the Hamiltonian constraint given by Eqn. (4.19) and disregarding the factor ordering ambiguity, we derive the following Wheeler de Witt wave equation:

$$\left[\frac{1}{4\Lambda} \frac{\partial^2}{\partial a^2} + \Lambda a^2 + (1 - A^2) \right] \Psi^{(AdS)}(a) = 0. \quad (4.21)$$

Inserting the complex coordinate transformation

$$a = \frac{1}{2\sqrt{\Lambda}} e^{i\pi/4} b$$

and

$$1 - A^2 \rightarrow i(1 - A^2),$$

in this equation, we obtain

$$\left(\frac{\partial^2}{\partial b^2} - \frac{1}{4} b^2 + (A^2 - 1) \right) \Psi^{(AdS)}(b) = 0. \quad (4.22)$$

Linearly dependent solutions to Eqn. (4.22) can also be given in terms of the parabolic cylinder functions D . They are [25]

$$\Psi_{\pm}^{(AdS)}(a) = D_{i(A^2-1)-1/2} \left(\pm 2\sqrt{\Lambda} a e^{-i\pi/4} \right), \quad (4.23)$$

and their complex conjugate counter-parts $(\Psi_{\pm}^{(AdS)}(a))^*$. By using the Airy-function representation of the parabolic cylindric functions [24,25], it can be seen that all these wave functions satisfy the reasonable boundary conditions $\lim_{a \rightarrow 0} \Psi_a \rightarrow 0$, and $\lim_{a \rightarrow \infty} \Psi_a \rightarrow 0$.

Following the procedure leading to Eqn. (3.17), we can now construct the pure quantum state that corresponds to the single AdS brane-world instanton. Thus, denoting $z = 2\sqrt{\Lambda} a \exp(-i\pi/4)$, we obtain from Eqn. (4.23) the state

$$\begin{aligned} & \Psi^{(AdS)(b)}(z) \\ &= \frac{1}{2} \int_{z_0}^{\infty} dz z D_{i(A^2-1)+1/2}(z) + D_{i(A^2-1)+1/2}(z) \Big|_{z_0}^{\infty}. \end{aligned} \quad (4.24)$$

If we now allow A^2 to be a complex quantity, $A^2 = 1 - iA_c^2$, and let A_c^2 to continuously vary from 0 to ∞ , we would again obtain a continuous spectrum with the ground state at $A_c^2 = 1/2$ given by $\Psi_0^{(AdS)(b)} \propto \exp(i\Lambda a_0^2)$.

V. PRIMORDIAL COSMOLOGICAL EVOLUTION

The evolution of the branes after creation would follow the same general pattern for all particular solutions (all particular five-dimensional equations of state) considered in the precedent two sections. It will always be given by analytically continuing metric (4.2) to real time. Expliciting the metric on the unit four-sphere as $d\Omega_4^2 = d\xi^2 + \sin^2 \xi d\Omega_3^2$, where $d\Omega_3^2$ is the metric on the unit three-sphere, we take ξ as the coordinate playing

the role of imaginary time (recall that τ is a radial extra coordinate in the five-dimensional manifold). Assuming for the analytical continuation $\xi \rightarrow iHt + \pi/2$, with $H = 1/a(\tau_0)$, metric (4.2) becomes

$$ds^2 = d\tau^2 + a(\tau)^2 [-H^2 dt^2 + \cosh^2(Ht) d\Omega_3^2]. \quad (5.1)$$

Every contributing brane will therefore inflate with a characteristic Hubble rate H which distinguishes solutions from each other. On the brane hypersurfaces at $\tau = \tau_0$, metric (5.1) reduces exactly to the line element of a four-dimensional de Sitter metric for each of the different Hubble rates. Metric (5.1) does not cover the whole spacetime but only the region from the brane position to the horizon at $\tau = 0$. The remaining region beyond $\tau = 0$, which describes the maximal extension of the corresponding baby universe spacetime [27,31], is obtained by analytically continuing $\tau \rightarrow ir$ and $Ht \rightarrow H\varphi - i\pi/2$, that is

$$ds^2 = dr^2 - a(r)^2 [H^2 d\varphi^2 + \sinh^2(H\varphi) d\Omega_3^2]. \quad (5.2)$$

The difference in Hubble rate of the distinct contributing solutions amounts to the feature that such solutions can be distinguished from each other by the nature of the inflationary process driven on them. It is worth noticing by instance that for the five-dimensional case with $\omega = -3/4$ it can be obtained that inflation will occur on the brane with a Hubble parameter given by

$$H_{\pm} = \frac{1}{2} \left[A^2 \pm \sqrt{A^4 - 4\Lambda \left(1 - \frac{\sigma_{\pm}^2}{\sigma_c^2} \right)} \right], \quad (5.3)$$

with the upper sign $+$ for the brane at $\tau > 0$ and the lower sign $-$ for the antibrane at $\tau < 0$. This parameter becomes that of the purely AdS space in the limit $A^2 \rightarrow 0$. Assuming that there is no dilatonic scalar field with nonzero effective potential [32], if $|\sigma_{\pm}| \geq \sigma_c$, we obtain that there will be inflation only on the brane, but not on the antibrane. On the contrary, if $|\sigma_{\pm}| < \sigma_c$, then there will be inflation both on the brane and the antibrane.

On the other hand, once the brane (or antibrane) hypersurface is fixed, all the initially contributing five-dimensional instantons considered in this work describe creation of a de Sitter space and can therefore be regarded to be contributing semiclassical paths for the creation of the universe. Now, what about the creation of a four-dimensional quintessence field in the brane?. Since the five-dimensional quintessence field ϕ was assumed to depend only on the extra coordinate τ (or r), the evolution of that field in the bulk becomes frozen on the brane with metric $ds^2 = -dt^2 + \cosh^2(Ht) d\Omega_3^2$. We contend that there may be essentially two ways for the creation of four-dimensional quintessence fields in the inflating branes. Firstly, one may resort to cosmological perturbations on the three-sphere Ω_{ij} starting from the above metric, so that $\Omega_{ij} \rightarrow \Omega_{ij} + \epsilon_{ij}$, with the perturbations ϵ_{ij} being expanded in terms of the usual tensor, vector (pure

gauge) and scalar harmonics, and the scalar field ϕ being perturbed so that $\phi \rightarrow \varphi(t) = \phi(\tau_0) + \delta(t)$, in which $\phi(\tau_0)$ is the frozen constant and $\delta(t) = \sum_n f_n(t) Q^{(n)}$, with the coefficients $f_n(t)$ depending on the Lorentzian time entering the four-dimensional metric and $Q^{(n)}$ the scalar harmonics (note that n simply denotes the set of labels n, l, m). If we next assume the perturbed field $\varphi(t)$ to satisfy a definition which is given by the four-dimensional counterpart of Eqns. (2.9) and (2.10) for the potential $V[\varphi(t)]$, and a conservation law $\rho_\varphi = \rho_{\varphi 0} R^{-3(1+\omega_\varphi(t))}$, with R the scale factor and $\omega_\varphi(t) = p_\varphi/\rho_\varphi > -1$, then we had created an usual four-dimensional quintessence tracking scenario on the brane world from our original model. On the other hand, besides other brane models in which the extra radial coordinate decreases from a very large initial value during cosmological evolution [33], it is also possible to envisage an alternative cosmological model derived from the results of the present work where the coordinate τ_0 on the brane (or antibrane) hypersurface increases from a minimal small initial value to follow the same pattern of cosmological expansion as the observable dimensions. In fact, all the de Sitter universes which are created through semiclassical paths fixed by the different possible values of the state equation parameter ω in five dimensions can be characterized by a generic cosmological constant given by

$$\Lambda = \frac{3}{a(\tau_0)^2},$$

where $a(\tau_0)$ is the constant value of the scale factor for each solution at the given brane hypersurface. If we would allow τ_0 to be a variable, then the cosmological constant will be variable too and could be looked at like though it were originated by a cosmic dark field, or "effective" quintessence field $\bar{\phi}$ in four-dimensional space, with equation of state $\rho_{\bar{\phi}} = \bar{\omega} p_{\bar{\phi}}$ and energy density given by

$$\rho_{\bar{\phi}} = \rho_{\bar{\phi} 0} \left(\frac{R_0}{R} \right)^{3(1+\bar{\omega})} = \frac{\Gamma_0}{a(\tau_0)^2}, \quad (5.4)$$

where Γ_0 is a constant. We then have

$$\frac{R_0}{R} = \left(\frac{\Gamma_0}{a(\tau_0)^2} \right)^{1/[3(1+\bar{\omega})]}. \quad (5.5)$$

On the other hand, one can also fix a general expression for the potential of the so-generated four-dimensional quintessence field $\bar{\phi}$ to be

$$V(\bar{\phi}) = V_0 \left(\frac{R_0}{R} \right)^{3(1+\bar{\omega})} = \frac{\Gamma_0}{a(\tau_0)^2}, \quad (5.6)$$

where V_0 is a constant. It follows that the four-dimensional "effective" quintessence field must be proportional to τ_0 . Thus, for the purely AdS path in the regime where $\tau_0 \ll 1$, we have $V \propto 1/(\bar{\phi} - \bar{\phi}_0)^2$, which is an inverse power-law potential whose shape has already

been considered in the literature [34]. Similar limiting expression for the potential are obtained also for the other considered solutions.

The above tentative "effective" quintessence model can be considered as a possible realization of the idea that the cosmological quintessence field springs from the physics of extra dimensions [12]. In the present case, the initial five-dimensional quintessence field can, in turn, be regarded as the result of the application of a similar mechanism to the extra sixth coordinate of a six-dimensional space minimally coupled to a six-dimensional quintessence field, and so on.

VI. CONCLUSIONS AND FURTHER COMMENTS

This paper contains a classical and quantum treatment of a primordial five-dimensional Friedmann-Robertson-Walker spacetime which is endowed with a negative cosmological constant and a quintessence field with variable state equation. The former is thought of as being responsible for the connection with particle physics and the latter is assumed to account for the content of dark energy needed to justify present cosmological observations. In this framework, the observable four-dimensional universe results in the form of a brane world by using an instantonic procedure.

After briefly analysing current tracking models in the early cosmological evolution, an Euclidean formalism is developed in which five- and four-dimensional gravity are minimally coupled to a scalar, homogeneous quintessence field, in the presence of a negative cosmological constant. We have studied the possible interrelations among the obtained solutions to the field equations and constraints for different state equations of the quintessence field, both in the four- and five-dimensional frameworks, and found that such solutions belong to only two broad categories: they represent either asymptotically AdS wormhole spaces or pure AdS spaces. In this way, we have uncovered new asymptotically AdS wormhole spacetimes. Following then a cutting a pasting procedure, it has been possible to construct consistent brane worlds from the given instantonic solutions. We have seen that whenever a five-dimensional instantonic solution describes a wormhole, one can construct an infinite or finite string of branes or brane-antibrane pairs. At least in the cases where it is possible to get solutions in closed form, one can also obtain explicit expressions for the quantum state of both, the AdS and asymptotically AdS wormhole spaces, so as for the brane worlds constructed from these spacetimes.

Our study is based on the idea that the universe contains a tracking quintessence field component whose state equation, ω , may take on any value, only restricted by $-1 < \omega < +1$, during its very early quantum evolution. Therefore, the quantum state of the very early universe

should be contributed by the semiclassical or quantum paths that correspond to the Euclidean solutions associated with all possible values of ω . As these solutions grow up, they will approach a spacetime structure whose observable four-dimensional sections exactly match a de Sitter space and can give rise, thereby, to fully equivalent four-dimensional geometric structures of the resulting inflating brane worlds, even though their internal bulk are different. This may be regarded as a geometrical no hair theorem for the creation of the universe: no matter the initial conditions, the universe is always created as an inflating de Sitter space.

The contribution of each particular solution for a given value of ω can be estimated by using the semiclassical expression $\Gamma \propto (-S_E)$, where S_E may be taken to be the Euclidean action that corresponds to the non-linear generalization of the metric on the brane, $ds^2 = d\tau^2 + a(\tau)^2 \gamma_{\mu\nu} dx^\mu dx^\nu$, where $\gamma_{\mu\nu}$ is the four-dimensional metric and $a(\tau)$ is the given classical solution. The Euclidean action for $\omega = 0$ and $\omega = -1/2$ were calculated in Refs. [27] and [15], respectively, for the general case of two concentric branes fixed at particular values of the extra dimension [15], either both at $\tau > 0$ or both at $\tau < 0$. None of them is diverging for finite values of the brane positions. For the case of the generalized solution that correspond to two single concentric branes at τ_0 and τ_1 with $\omega = -3/4$, we obtain

$$S_E = \sum_{i=0}^1 (-1)^i \left\{ F\tau_i + \cosh\left(\sqrt{\Lambda}\tau_i\right) \left[G \sinh\left(\sqrt{\Lambda}\tau_i\right) + H \sinh^2\left(\sqrt{\Lambda}\tau_i\right) + I \sinh^3\left(\sqrt{\Lambda}\tau_i\right) + J \right] \right\} S_E^{(i)},$$

where

$$F = \frac{A^8 + \frac{3}{8}\alpha^2 - 3A^4\alpha}{4\Lambda^{3/2}a_i^2}$$

$$G = \frac{3(2A^4 - \alpha/8)}{4\Lambda^2 a_i^2}$$

$$H = \frac{13A^2\sqrt{\alpha}}{12\Lambda^2 a_i^2}$$

$$I = \frac{5\alpha}{16\Lambda^2 a_i^2}$$

$$J = \frac{A^2(5A^4 - 8\alpha/3)}{8\Lambda^3 a_i^2},$$

with $a_i \equiv a(\tau_i)$ and

$$S_E^{(i)} = -\frac{V_4^{(\gamma)}}{4\pi G_N H_i^2},$$

in which $V_4^{(\gamma)}$ is the dimensionless volume of the manifold with metric $\gamma_{\mu\nu}$, $G_N = G_5 \Lambda^{3/2}/\alpha$ is the Newton constant and $H_i = a_i^{-1}$ is the Hubble constant at the brane at τ_i . Again this action gives nonvanishing contributions for finite values of τ_i . Following the procedure described in Ref. [27], one can also compute the nucleation probability for a finite or infinite string of brane-antibrane pairs for solutions which represent asymptotically AdS wormholes. This gives convergent nonzero contributions for all cases, provided S_E is nonvanishing. Such solutions are, moreover, nonsingular as the Weyl tensor constructed from them is always finite, even for single branes, a result which is no longer valid for the case that the solution describes a pure AdS spacetime [15].

On the other hand, all the solutions considered in this paper can be related with a ground state for gravity on the corresponding brane. This state and the corresponding Kaluza-Klein spectrum should be obtained by solving the equations of motion for the metric perturbations expressed in terms of the conformal time [14,15]. In all the cases, the normalizable ground state corresponding to the massless graviton is proportional to $a^{3/2}$ and the spectrum of Kaluza-Klein excitations is continuous and separated by a gap from the zero mode [15]. We shall consider these issues quite more in detail in a future publication.

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- [1] S. Perlmutter et al., *Astrophys. J.* 517, 565 (1999); A.G. Riess et al. *Astron. J.* 116, 1009 (1998); I. Zehavi and A. Dekel, *Nature* 401, 252 (1999).
 - [2] A.G. Riess et al. *Support for an accelerating universe and a glimpse of the epoch of deceleration*, astro-ph/0104455, *Astrophys. J.* (in press).
 - [3] P. de Bernardis et al. *Nature*, 404, 955 (2000).
 - [4] S. Hanany et al. *Astrophys. J.* 545, 1 (2000).
 - [5] Information can be found in <http://snap.lbl.gov> See also Y. Wang, *Astrophys. J.* 531, 676 (2000); J.A. Tyson, D. Wittman and J.R.P. Angel, astro-ph/0005381; M.S. Turner and J.A. Tyson, *Rev. Mod. Phys.* 71, S145 (1999).
 - [6] Information can be found in <http://map.gsfc.nasa.gov/>, and <http://astro.estec.esa.nl/SA-general/Projects/Planck/>

- [7] N. Bahcall, J.P. Ostriker, S. Perlmutter and P.J. Steinhardt, *Science* 284, 1481 (1999).
- [8] C. Wetterich, *Nucl. Phys.* B302, 668 (1988); J.C. Jackson and M. Dodgson, *Mon. Not. R. Astron. Soc.* 297, 923 (1998); J.C. Jackson, *Mon. Not. R. Astron. Soc.* 296, 619 (1998); R.R. Caldwell, R. Dave and P.J. Steinhardt, *Phys. Rev. Lett.* 80, 1582 (1998); L. Wang and P.J. Steinhardt, *Astrophys. J.* 508, 483 (1998); R.R. Caldwell and P.J. Steinhardt, *Phys. Rev. D* 57, 6057 (1998); G. Huey, L. Wang, R. Dave, R.R. Caldwell and P.J. Steinhardt, *Phys. Rev. D* 59, 063005 (1999).
- [9] A. Balbi, C. Baccigalupi, S. Matarrese, F. Perrota and N. Vittorio, *Astrophys. J.* 547, 289 (2001); P. Brax, J. Martin and A. Riazuelo, *Phys. Rev. D* 62, 103505 (2000); L. Wang, R.R. Caldwell, J.P. Ostriker and P.J. Steinhardt, *Astrophys. J.* 530, 17 (2000).
- [10] I. Zlatev, L. Wang and P.J. Steinhardt, *Phys. Rev. Lett.* 82, 896 (1999); *ibid*, *Phys. Rev. D* 59, 123504 (1999); I. Zlatev and P.J. Steinhardt, *Phys. Lett. B* 459, 570 (1999).
- [11] P.J. Steinhardt, in *Critical problems in physics*, edited by V.L. Fitch and D.R. Marlow (Princeton Univ. Press, Princeton, USA, 1997).
- [12] P.F. González-Díaz, *Phys. Lett. B* 481, 353 (2000); K.I. Izawa, *Prog. Theor. Phys.* 105, 185 (2001); K.I. Maeda, *Brane quintessence*, astro-ph/0012313; J.P. Ostriker and P.J. Steinhardt, *Sci. Am.* January 2001, pag. 36.
- [13] K. Akama; *Lect. Notes Phys.* 176, 267 (1982) V. Rubakov and M. Shaposhnikov, *Phys. Lett. B* 125, 136 (1983); M. Visser, *Phys. Lett. B* 159, 22 (1985); N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett. B* 429, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett. B* 436, 257 (1998); L. Anchordoqui, C. Nuñez and K. Olsen *JHEP* 001, 050 (2000); P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, *Phys. Lett. B* 477, 285 (2000); P. Binétruy, C. Deffayet and D. Langlois, *Nucl. Phys. B* 565, 269 (2000); T. Shiromizu, K.I. Maeda and M. Sasaki, *Phys. Rev. D* 62, 024012 (2000); J. Garriga and T. Tanaka, *Phys. Rev. Lett.* 84, 2778 (2000), A. Chamblin, S.W. Hawking and H.S. Reall, *Phys. Rev. D* 61, 065007 (2000), N. Kaloper, *Phys. Rev. D* 60, 123506 (1999); C. Barceló and M. Visser, hep-th/0009032, *JHEP* 0010, 019 (2000); C. Csáki, M.L. Graesser and G.D. Kribs, *Phys. Rev. D* 63, 065002 (2001). For extensive lists of relevant references, see these two latter papers.
- [14] L. Randall and R. Sundrum, *Phys. Rev. Lett.* 83, 3370; 4690 (1999).
- [15] J. Garriga and M. Sasaki, *Phys. Rev. D* 62, 043523 (2000).
- [16] A. Vilenkin, *Phys. Rev. D* 58, 067301 (1998).
- [17] J.B. Hartle and S.W. Hawking, *Phys. Rev. D* 28, 2960 (1983).
- [18] S.W. Hawking and D.N. Page, *Phys. Rev. D* 42, 2655 (1990).
- [19] P.J. Steinhardt, L. Wang and I. Zlatev, *Phys. Rev. D* 59, 123504 (1999).
- [20] J. M. Maldacena, *Adv. Theor. Math. Phys.* 2, 231 (1998).
- [21] P.F. González-Díaz, *Phys. Rev. D* 62, 023513 (2000).
- [22] C. Barceló, L.J. Garay, P.F. González-Díaz and G.A. Mena Marugán, *Phys. Rev. D* 53, 3162 (1996).
- [23] P.F. González-Díaz, *Nucl. Phys. B* 351, 767 (1991).
- [24] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, USA, 1965).
- [25] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integral, Series, and Products* (Academic Press, Inc., Orlando, USA, 1980).
- [26] W. Israel, *N. Cimento B* 44, 1 (1966); B48, 463 (1966).
- [27] P.F. González-Díaz, *Brane-worlds in quantum cosmology*, to be published.
- [28] S.W. Hawking and R. Penrose, *The nature of space and time* (Princeton Univ. Press, Princeton, USA, 1996).
- [29] P.F. González-Díaz, *Elliptic and circular wormholes*, gr-qc/9306031
- [30] J.J. Halliwell and R. Laflamme, *Class. Quantum Grav.* 6, 1838 (1989).
- [31] P.F. González-Díaz, *Brane russian doll*, hep-th/0105088, *Phys. Lett. B* (in press).
- [32] Y. Himemoto and M. Sasaki, *Phys. Rev. D* 63, 044015 (2001).
- [33] P.F. González-Díaz, *Phys. Lett. B* 486, 158 (2000).
- [34] B. Ratra and P.J.E. Peebles, *Phys. Rev. D* 37, 3406 (1988); P.E.J. Peebles and B. Ratra, *Astrophys. J. Lett.* 325, L17 (1988).